MATH 3235 Probability Theory

2/21/23

If X is a r.v.

$$F(x) = \mathbb{P}(X \leq x)$$

F is c.d.f. of X

$X : \Omega \to \mathbb{R}$ such that

$$\{\omega \mid X(\omega) \leq x\} \in \mathcal{F}$$

$$X^{-1}((-\infty, x]) \in \mathcal{F}$$

This is a formal definition of r.v.

If X is discrete than

$F(x)$ is constant but for

a countable number of jumps.

If X is an integer valued r.v.

$$P_X(x) = F_X(x) - F_X(x-1)$$

If x_k are the possible value of X, ordered. Then

$$P_X(x_k) = F_X(x_k) - F(x_{k-1})$$

$$F_X(x_k) = \sum_{x_j \leq x_k} P_X(x_j)$$

$$F_X(x) = \sum_{x_j \leq x} P_X(x_j)$$

$X : \Omega \to \mathbb{R}$

X is a r.v. whose image is $\{x_k\}_{k=1,\dots,\infty}$ $\quad P_X(x_k) = p_k$

$\Omega = \{x_k\}_{k=1,\dots,\infty}$ $\quad \mathbb{P}(x_k) = p_k$

X : identity on Ω.

Continuous.

$$\mathbb{P}(a < X \le b) = \int_a^b f_X(x)\,dx$$

f_X is called p.d.f.

$$\mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(y)\,dy = F(x)$$

$$\Downarrow$$

$$F'(x) = f(x)$$

We will call a r.v. continuous if $F'(x)$ exists, but for a finite number of point.

$[0,1]$ I want to pick a point at random uniformly in $[0,1]$.

$$\mathbb{P}(0 \le X \le 1) = 1$$

$F(x) = 0 \qquad x < 0$

$F(x) = 1 \qquad x > 1$

$a < 1$

$\mathbb{P}(0 \le X \le a) = a$

$0 < a < b < 1$

$\mathbb{P}(a \le X \le b) = b - a$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

$f_X(x) = F'(x)$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

2) X is uniform in $[A, B]$

$A < B$

p.d.f.

$$f_X(x) = \begin{cases} C & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$1 = \int_A^B C\,dx = C(B-A)$$

$$C = \frac{1}{B-A}$$

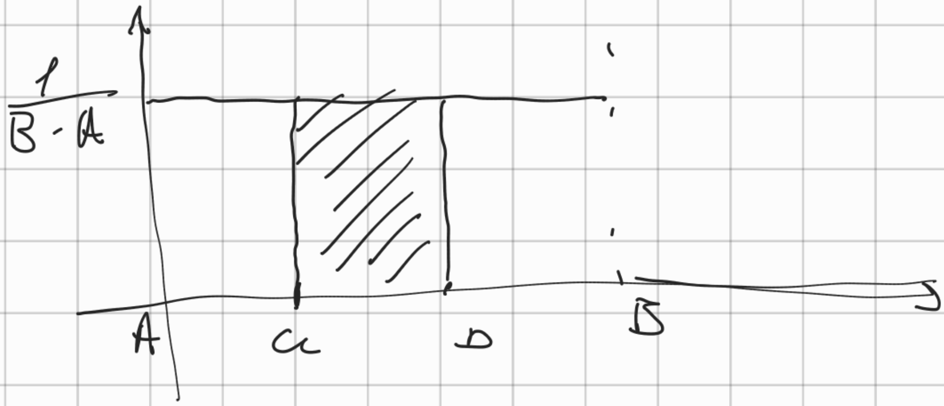
$$f_X(x) = \begin{cases} \frac{1}{B-A} & A \leq x \leq B \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 & x > B \\ \frac{x}{B-A} & A \leq x \leq B \\ 0 & x < A \end{cases}$$

$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)\,dx$$

$$= \frac{b-a}{B-A}$$

if $A \le a < b \le B$.



In general!

$\mathbb{P}(a \le X \le b)$ is the area under the graph of $f_X(x)$ between a and b.

$$\mathbb{P}(a \le X \le b) = F(b) - F(a) =$$

$$= \frac{b-a}{B-A}$$

$$\mathbb{P}(x \le X \le x + dx) = f(x)dx$$

dx is very small.

$$\lim_{h \to 0} \frac{\mathbb{P}(x \le X \le x + h)}{h} = f(x)$$

Geometric r.v.

$$P_X(n) = (1-p)^{n-1} p$$

τ $\frac{1}{N}$ assume That $p = \frac{\lambda}{N}$

$$\mathbb{P}(X \le n) - (1-p)^n$$

n is The number of unit of length $\frac{1}{N}$ before The first arrival.

Arrival Time T in seconds

$$\mathbb{P}(T \ge t) = \left(1 - \frac{\lambda}{N}\right)^{tN}$$

If I take the limit for $N \to \infty$

I get

$$\mathbb{P}(T \geq t) = \left(\left(1 - \frac{\lambda}{N}\right)^{N}\right)^{t}$$

$$\underset{N\to\infty}{=} e^{-\lambda t}$$

$$\mathbb{P}(T \leq t) = 1 - e^{-\lambda t} = F_T(t)$$

$$f(t) = \lambda e^{-\lambda t}$$

Esponential r.v. with par λ

(Warning: some authors write

$$f(t) = \frac{1}{\theta} e^{-\frac{t}{\theta}}$$

)

$$\mathbb{P}(T \geq t+s \mid T \geq t) =$$

$$\frac{\mathbb{P}(T \geq t+s \ \& \ \cancel{T \geq t})}{\mathbb{P}(T \geq t)} =$$

$$\frac{\mathbb{P}(T \geq t+s)}{\mathbb{P}(T \geq t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$\mathbb{P}(T \geq t+s \mid T \geq t) = \mathbb{P}(T \geq s)$$

$A = \{T \geq t+s\}$ $\quad$ $B = \{T \geq t\}$

$A \subset B$ $\quad$ $A \cap B = A$

Lack of Memory.

Normal r.v.

Standard Normal r.v.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$\Phi(x)$ probability integral.

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi}$$

$$\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx\right)^2 = 2\pi$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}dy = 2\pi$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}}dx\,dy = 2\pi$$

$$\int_{-\pi}^{\pi}d\theta \int_{0}^{\infty}d\rho\,\rho\,e^{-\frac{\rho^2}{2}} = 2\pi\int_{0}^{\infty}\rho\,e^{-\frac{\rho^2}{2}}d\rho$$

$$= 2\pi\left(-e^{-\frac{\rho^2}{2}}\right)\Big|_{0}^{\infty} = 2\pi$$